

Exam 4 Outline

Multiple Integration

- A. Integration in Two Variables (Rectangular Domains)
- B. Double Integrals over More General Regions
- C. Polar Coordinates and Double Integrals
- D. Triple Integrals
- E. Cylindrical and Spherical Coordinates

Exam 4 Formula Sheet

- For a regular curve $\vec{r}(t)$,

$$- \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$- \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

$$- \vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

- For a regular curve $\vec{r}(t)$, the curvature is given by

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}.$$

- **The Second Derivative Test for Functions of Two Variables**

Let $f(x, y)$ be a twice differentiable function, and assume its second partial derivatives are continuous. Let (a, b) be a critical point for f , and define the Hessian of f at (a, b) to be

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

- If $D(a, b) < 0$, then (a, b) is a Saddle Point.
- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a Local Minimum.
- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a Local Maximum.
- If $D(a, b) = 0$, the test is inconclusive.

- **Lagrange Multipliers**

If a differentiable function f has a local maximum or local minimum on a constraint curve of the form $g = \text{constant}$ (where g is a differentiable function), then there is a constant λ so that

$$\nabla f = \lambda \nabla g$$

at the local extremum (so long as ∇g is not zero at the point in question).

- **Polar Coordinates**

- $x = r \cos \theta, \quad y = r \sin \theta$
- $r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}$
- $dA = r \, dr \, d\theta$

- **Cylindrical Coordinates**

- $x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$
- $r = \sqrt{x^2 + y^2}, \quad \tan \theta = \frac{y}{x}, \quad z = z$
- $dV = r \, dr \, d\theta \, dz$

- **Spherical Coordinates**

- $x = \rho \cos \theta \sin \varphi$, $y = \rho \sin \theta \sin \varphi$, $z = \rho \cos \varphi$
- $\rho = \sqrt{x^2 + y^2 + z^2}$, $\tan \theta = \frac{y}{x}$, $\varphi = \arccos(z/\rho)$
- $dV = \rho^2 \sin \varphi \, d\rho d\theta d\varphi$

- **Change of Variables in Multiple Integrals**

Suppose a transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is one-to-one (except perhaps on the boundary of \mathcal{D}), maps a region \mathcal{D} in the uv -plane to a region \mathcal{C} in the xy -plane, and g and h have continuous partial derivatives in a region containing \mathcal{D} . If f is a continuous function over the region \mathcal{C} , then

$$\iint_{\mathcal{C}} f(x, y) dx dy = \iint_{\mathcal{D}} f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv,$$

where the Jacobian of the transformation is given by

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

The analogous statement holds for triple integrals.